Conservation laws and globality for the variation of local Noether strong currents

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Invariant variational problems

- we analyze the interplay of inverse variational problems with invariance properties
- we first review some cohomological aspects concerned with local variational problems, investigating when local problems are equivalent to global ones
- then study Noether currents associated with symmetries of global locally variational dynamical forms (generalized symmetries)

Geometric framework

- modern geometrical formulations of the calculus of variations on fibered manifolds include a large class of theories for which the Euler–Lagrange operator is a morphism of an exact sequence
- indeed, the module in degree $(n+1)$, 'contains' dynamical forms
- dynamical forms which are only *locally variational*, *i.e.* which are closed in the differential complex and define a non trivial cohomology class, admit a system of local Lagrangians
- Lagrangians are invariant w.r.t. generalized symmetries only up to a horizontal differential (a divergence) (Noether–Bessel-Hagen theorem)

Geometric framework

- Noether theorems can be formulated by the *variational Lie derivative*
- variations can be recognized within this picture
- considering the cohomology defined by a system of local Lagrangians, investigate under which conditions the variational Lie derivative of associated local strong Noether currents is equivalent to a system of global and conserved currents
- in particular, variation vector fields -as generalized symmetries belonging to the kernel of the second variational derivative of the local problem- generate canonical Noether currents - associated with variations of local Lagrangians - which turn out to be conserved *along any section*
- we characterize the variation of canonical Noether currents associated with a local variational problem

Variational sequence

- the variational sequence defined on a fibered manifold $\pi : \boldsymbol{Y} \to \boldsymbol{X}$, with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n + m$.
- for $r \geq 0$ the r-jet space $J^r\boldsymbol{Y}$ of jet prolongations of sections of the fibered manifold π
  - the natural fiberings $\pi^r_s : J^r\boldsymbol{Y} \to J^s\boldsymbol{Y}$, $r \geq s$, and $\pi^r : J^r\boldsymbol{Y} \to \boldsymbol{X}$
  - among these the fiberings π^r_{r-1} are *affine bundles* which induce the natural fibered splitting

$$J^r\boldsymbol{Y} \times_{J^{r-1}\boldsymbol{Y}} T^*J^{r-1}\boldsymbol{Y} \simeq J^r\boldsymbol{Y} \times_{J^{r-1}\boldsymbol{Y}} (T^*\boldsymbol{X} \oplus V^*J^{r-1}\boldsymbol{Y})$$

- decomposition of the exterior differential on $\boldsymbol{Y}$ in the *horizontal* and *vertical differential*, $(\pi^{r+1}_r)^* \circ d = d_H + d_V$
- $(j^r\Xi, \xi)$ jet prolongation of a *projectable vector field* (Ξ, ξ) on $\boldsymbol{Y}$, and $j^r\Xi_H$ and $j^r\Xi_V$ the horizontal and the vertical part of $j^r\Xi$, respectively.

Variational sequence

- let ρ be a q-form on $J^r\boldsymbol{Y}$:
  natural decomposition of the pull-back by the affine projections of ρ:

$$(\pi^{r+1}_r)^*\rho = \sum_{i=0}^{q} p_i\rho\,,$$

  where $p_i\rho$ is the i-contact component of ρ
- starting from this splitting one can define
  sheaves of contact forms Θ^*_r, suitably characterized by the kernel of conveniently chosen p_i
- the sheaves Θ^*_r form an exact subsequence of the de Rham sequence on $J^r\boldsymbol{Y}$ and one can define the quotient sequence

$$0 \to \boldsymbol{R_Y} \to \cdots \to^{\mathcal{E}_{n-1}} \Lambda^n_r/\Theta^n_r \to^{\mathcal{E}_n} \Lambda^{n+1}_r/\Theta^{n+1}_r \to^{\mathcal{E}_{n+1}} \Lambda^{n+2}_r/\Theta^{n+2}_r \to^{\mathcal{E}_{n+2}} \cdots \to 0$$

- the *r–th order variational sequence* over the fibered manifold $\boldsymbol{Y} \to \boldsymbol{X}$ is a soft sheaf resolution of the constant sheaf $\boldsymbol{R_Y}$ over $\boldsymbol{Y}$

Variational sequence

- the quotient sheaves (the sections of which are classes of forms modulo contact forms) in the variational sequence can be represented as sheaves $\mathcal{V}^k_r$ of k-forms on jet spaces of higher order:
- currents are sheaf sections ϵ of $\mathcal{V}^{n-1}_r$ and the quotient morphism $\mathcal{E}_{n-1} \simeq d_H$ is represented by a total divergence
- Lagrangians are sections λ of $\mathcal{V}^n_r$, while $\mathcal{E}_n$ is called the Euler-Lagrange morphism
- sections η of $\mathcal{V}^{n+1}_r$ are called *dynamical forms* or also *source forms*, while $\mathcal{E}_{n+1}$ is called the Helmholtz morphism

Variational cohomology

- denote by $H^*_{\mathrm{VS}}(\boldsymbol{Y})$ the cohomology groups of the corresponding complex of global sections

$$0 \longrightarrow \boldsymbol{R_Y} \longrightarrow \dots \xrightarrow{\mathcal{E}_{n-1}} (\Lambda^n_r/\Theta^n_r)_{\boldsymbol{Y}} \xrightarrow{\mathcal{E}_n} (\Lambda^{n+1}_r/\Theta^{n+1}_r)_{\boldsymbol{Y}} \xrightarrow{\mathcal{E}_{n+1}} (\Lambda^{n+2}_r/\Theta^{n+2}_r)_{\boldsymbol{Y}} \xrightarrow{\mathcal{E}_{n+2}} \dots \xrightarrow{d} 0$$

- **D. Krupka** ($\sim$ 1990) stated the following important Theorem:

  - the variational sequence is a soft sheaf resolution of the constant sheaf $\boldsymbol{R_Y}$ over $\boldsymbol{Y}$

  - hence the cohomology of the complex of global sections is naturally isomorphic to both the Čech cohomology of $\boldsymbol{Y}$ with coefficients in the constant sheaf $\boldsymbol{R_Y}$ and the de Rham cohomology $H^k_{\mathrm{dR}}\boldsymbol{Y}$

Variational cohomology

- let now $\boldsymbol{K}_r^p \equiv \text{Ker } \mathcal{E}_p$
  $\rightarrow$ we have the short exact sequence of sheaves

$$0 \longrightarrow \boldsymbol{K}_r^p \stackrel{i}{\longrightarrow} \mathcal{V}_r^p \stackrel{\mathcal{E}_p}{\longrightarrow} \mathcal{E}_p(\mathcal{V}_r^p) \longrightarrow 0$$

- in particular $\mathcal{E}_n(\mathcal{V}_r^n)$ is the sheaf of Euler–Lagrange forms:
  for a global section $\eta \in (\mathcal{V}_r^{n+1})_{\boldsymbol{Y}}$ we have
  $\eta \in (\mathcal{E}_n(\mathcal{V}_r^n))_{\boldsymbol{Y}}$ if and only if $\mathcal{E}_{n+1}(\eta) = 0$,
  which are the Helmholtz conditions of local variationality

Variational cohomology

- the above gives rise to a well known diagram of cochain complexes:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C^0(Y, K^p_r) \arrow[r, "i"] \arrow[d, "\mathfrak{d}"] & C^0(Y, \mathcal{V}^p_r) \arrow[r, "\mathcal{E}_p"] \arrow[d, "\mathfrak{d}"] & C^0(Y, \mathcal{E}_p(\mathcal{V}^p_r)) \arrow[r] \arrow[d, "\mathfrak{d}"] & 0 \\
0 \arrow[r] & C^1(Y, K^p_r) \arrow[r, "i"] \arrow[d] & C^1(Y, \mathcal{V}^p_r) \arrow[r, "\mathcal{E}_p"] \arrow[d] & C^1(Y, \mathcal{E}_p(\mathcal{V}^p_r)) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

whereby, indeed, we recognize the *connecting homomorphism* $\delta_p = i^{-1} \circ \mathfrak{d} \circ \mathcal{E}_p^{-1}$ as a mapping of cohomologies (here $\mathfrak{d}$ is the *coboundary operator*).

Variational cohomology

- the exact sequence above gives rise to the long exact sequence in Čech cohomology

$$0 \longrightarrow (\boldsymbol{K}_r^p)_{\boldsymbol{Y}} \longrightarrow (\mathcal{V}_r^p)_{\boldsymbol{Y}} \longrightarrow (\mathcal{E}_p(\mathcal{V}_r^p))_{\boldsymbol{Y}} \xrightarrow{\delta_p} H^1(\boldsymbol{Y}, \boldsymbol{K}_r^p) \longrightarrow 0$$

- hence, every $\eta \in (\mathcal{E}_n(\mathcal{V}_r^n))_{\boldsymbol{Y}}$ (*i.e.* locally variational) defines a cohomology class $\delta_n(\eta) \in H^1(\boldsymbol{Y}, \boldsymbol{K}_r^n) \simeq H_{VS}^{n+1}(\boldsymbol{Y}) \simeq H_{dR}^{n+1}(\boldsymbol{Y})$
- furthermore, every $\mu \in (d_H(\mathcal{V}_r^{n-1}))_{\boldsymbol{Y}}$ (*i.e.* variationally trivial) defines a cohomology class $\delta_{n-1}(\mu) \in H^1(\boldsymbol{Y}, \boldsymbol{K}_r^{n-1}) \simeq H_{VS}^n(\boldsymbol{Y}) \simeq H_{dR}^n(\boldsymbol{Y})$

Global locally variational equations

- note that η is globally variational if and only if $\delta_n(\eta) = 0$.
- we are now interested to the nontrivial case where

$$\mathcal{E}_{n+1}(\eta) = 0 \qquad \mathfrak{d}\eta = 0$$
$$\delta_n(\eta) \neq 0 \implies \mathfrak{d}\lambda_i \neq 0$$

  whereby $\eta = \mathcal{E}_n(\lambda)$ can be solved only locally, *i.e.* for any countable good covering of $\boldsymbol{Y}$ there exists a local Lagrangian λ_i over each subset $\boldsymbol{U}_i \subset \boldsymbol{Y}$ such that $\eta_i = \mathcal{E}_n(\lambda_i)$.
- a system of local sections λ_i of $(\mathcal{V}^n_r)_{U_i}$ such that $\mathcal{E}_n((\lambda_i - \lambda_j)|_{U_i \cap U_j}) = 0$, is what we call a *local variational problem*; every nontrivial cohomology class gives rise to local variational problems

Variational Lie derivative and Noether theorems

- formulate the Noether Theorems in terms of variational Lie derivatives of classes of forms modulo the contact structure
- the case $q \leq n-1$

Theorem

Let $\alpha \in \mathcal{V}_r^q$, $0 \leq q \leq n-1$, and let Ξ be a π-projectable vector field on $\boldsymbol{Y}$; the following holds locally

$$\mathcal{L}_{\Xi}\alpha = \Xi_H \rfloor \mathcal{E}_q(\alpha) + \mathcal{E}_{q-1}(\Xi_V \rfloor \tilde{p}_{d_V\alpha} + \Xi_H \rfloor \alpha)$$

Here $\tilde{p}$ is a generalized momentum associated with an horizontal p-form

Variational Lie derivative and Noether theorems

- the case $q = n$ Noether Theorem I

Theorem

Let $\alpha \in \mathcal{V}^n_r$ and Ξ be a π-projectable vector field on $\boldsymbol{Y}$; the following holds (locally):

$$\mathcal{L}_{\Xi}\alpha = \Xi_V \rfloor \mathcal{E}_n(\alpha) + \mathcal{E}_{n-1}(\Xi_V \rfloor p_{d_V \alpha} + \Xi_H \rfloor \alpha)$$

Theorem

Let $q = n + k$, with $k \geq 1$ and $\alpha \in \mathcal{V}^q_r$. Let Ξ be a π-projectable vector field on $\boldsymbol{Y}$; we have

$$\mathcal{L}_{\Xi}\alpha = \Xi_V \rfloor \mathcal{E}_q(\alpha) + \mathcal{E}_{q-1}(\Xi_V \rfloor \alpha)$$

- the case $q = n + 1$ for locally variational dynamical forms encompasses Noether Theorem II, describing Noether–Bessel-Hagen symmetries

Local and global Noether–Bessel-Hagen currents

- let now η_λ be the Euler–Lagrange morphism of a local variational problem and let $j^r\Xi$ be a generalized symmetry
- the Noether Theorems imply that
  $0 = \Xi_V \rfloor \eta_\lambda + d_H(\epsilon_i - \beta_i)$,
  where $\epsilon_i \equiv \epsilon(\lambda_i, \Xi) = j^r\Xi_V \rfloor p_{d_V\lambda_i} + \xi \rfloor \lambda_i$
  is called the *canonical* Noether current
- under the assumption of a nontrivial cohomology, the Noether–Bessel-Hagen current $\epsilon_i - \beta_i$ is a *local* object and it is conserved along the solutions of Euler–Lagrange equations (*local conservation law*)
- note that, since $0 = \mathcal{L}_{j^r\Xi}\eta = \mathcal{E}_n(\Xi_V \rfloor \eta_\lambda)$, the horizontal n-form $\Xi_V \rfloor \eta_\lambda$ defines a cohomology class and we have that the local currents are the restrictions of a global conserved current if and only if the cohomology class $\delta_{n-1}(\Xi_V \rfloor \eta_\lambda) \in H^n_{dR}(\boldsymbol{Y})$ vanishes

Local variational problems equivalent to global ones

- now note that as a consequence of linearity and naturality properties, the variational Lie derivative trivializes cohomology classes
- in particular, we have $\eta_{\mathcal{L}_\Xi \lambda_i} = \mathcal{E}_n(\Xi_V \rfloor \eta_\lambda) = \mathcal{L}_\Xi \eta_\lambda$
- this implies that $\delta_n(\mathcal{L}_\Xi \eta_\lambda) = \delta_n(\eta_{\mathcal{L}_\Xi \lambda_i}) = 0$, although we assumed $\delta_n(\eta_\lambda) \neq 0$
- thus Euler–Lagrange equations of the local problem defined by $\mathcal{L}_\Xi \lambda_i$ are equal to Euler–Lagrange equations of the global problem defined by the global object $\Xi_V \rfloor \eta_\lambda$
- an analogous result holds true also at any degree $k \leq n$ in the variational sequence; specifically for local potentials of variationally trivial classes of horizontal forms

Global variationally trivial horizontal forms

- let μ be a global variationally trivial Lagrangian, this means that we have a 0-cocycle of currents ν_i such that $\mu = d_H\nu_i$
- assume that these are strictly locally defined objects:

$$\mathcal{E}_n(\mu) = 0 \qquad \mathfrak{d}\mu_\nu = 0$$
$$\delta_{n-1}(\mu_\nu) \neq 0 \implies \mathfrak{d}\nu_i \neq 0$$

- consider the Lie derivative $\mathcal{L}_\Xi\nu_i$ and the corresponding $\mu_{\mathcal{L}_\Xi\nu_i}$
- we have $\mu_{\mathcal{L}_\Xi\nu_i} = d_H(\Xi_H\rfloor\mu_\nu + \Xi_V\rfloor p_{d_V\mu_\nu}) = \mathcal{L}_\Xi\mu_\nu$,
so that $\delta_{n-1}(\mathcal{L}_\Xi\mu_\nu) = 0$, although we assumed $\delta_{n-1}(\mu_\nu) \neq 0$
- the local problem defined by $\mathcal{L}_\Xi\nu_i$ is variationally equivalent to the global problem defined by the global object $\Xi_H\rfloor\mu_\nu + \Xi_V\rfloor p_{d_V\mu_\nu}$ (recall that $\mu_\nu = d_H\nu_i$ is assumed to satisfy $\mathfrak{d}\mu_\nu = 0$, *i.e.* it is a global object)

Global variationally trivial horizontal forms

Example

- *let Ξ be a generalized symmetry*
- *we have* $\mathcal{L}_{\Xi}\eta_{\lambda} = 0$
  then, in particular, $\delta_n(\mathcal{L}_{\Xi}\eta_{\lambda}) \equiv 0$
- *furthermore, under the same assumption, we have* $\mathcal{E}_n(\Xi_V \rfloor \eta_{\lambda}) = 0$
  then, if $\delta_{n-1}(\Xi_V \rfloor \eta_{\lambda}) \neq 0$*, there exists a* 0*-cocycle* ν_i *as above, defined by* $\mu_{\nu} = \Xi_V \rfloor \eta_{\lambda} \equiv d_H \nu_i$
- *in this case, divergence expressions of the local problem defined by* $\mathcal{L}_{\Xi}\nu_i$ *coincide with divergence expressions for the global current* $\epsilon_{d_H \nu_i} = \Xi_H \rfloor \Xi_V \rfloor \eta_{\lambda} + \Xi_V \rfloor p_{d_V(\Xi_V \rfloor \eta_{\lambda})}$

Canonical strong Noether currents

- we have that $\mathcal{L}_{j^r\Xi}\lambda_i = \Xi_V \rfloor \eta_{\lambda_i} + d_H\epsilon_{\lambda_i}$
- on the other hand $\mathcal{L}_{j^r\Xi}\eta = 0 \implies \mathcal{E}_n(\Xi_V \rfloor \eta_{\lambda_i}) = 0$,
- we saw that if $\delta_{n-1}(\Xi_V \rfloor \eta) \neq 0$, then $\Xi_V \rfloor \eta = d_H\nu_i$, with $\mathfrak{d}\nu_i \neq 0$,
- therefore the Noether theorems I and II for a local system of Lagrangians read

$$\mathcal{L}_{j^r\Xi}\lambda_i = d_H(\nu_i + \epsilon_i)$$

where $\nu_i + \epsilon_i$ is called a (local) canonical strong Noether current (note: if $\mathcal{L}_{j^r\Xi}\lambda_i = 0$ would hold true -not our case!-, then we would have a local Noether strong conservation law $d_H(\nu_i + \epsilon_i) = 0$)

- *under the conditions* $\delta_n(\eta) \neq 0$ and $\delta_{n-1}(\Xi_V \rfloor \eta_{\lambda_i}) \neq 0$ we necessarily have

$$\mathfrak{d}(\nu_i + \epsilon_i) \neq 0$$

Some technical results

- case of a generalized symmetry: by the Noether Th. (II), $\mathcal{L}_{j^r\Xi}\lambda_i =_{loc.} d_H\beta_i$
- suppose that $\mathfrak{d}\mathcal{L}_{j^r\Xi}\lambda_i = 0$ $(\mathcal{E}_n(\mathcal{L}_{j^r\Xi}\lambda_i) = \mathcal{E}_n(\Xi_V \rfloor \eta)$ implies that $\mathcal{L}_{j^r\Xi}\lambda_i$ can be seen as the local restriction of a global object modulo a horizontal differential)
- since $\mathcal{E}_n(\mathcal{L}_{j^r\Xi}\lambda_i) = 0$ then $\mathcal{L}_{j^r\Xi}\lambda_i$ defines a cohomology class
- if $\delta_{n-1}(\mathcal{L}_{j^r\Xi}\lambda_i) \neq 0$ then $\mathfrak{d}\beta_i \neq 0$
- notice that if $\mathfrak{d}\mathcal{L}_{j^r\Xi}\lambda_i = 0$ then $0 = \mathfrak{d}d_H\beta_i = d_H\mathfrak{d}\beta_i$, *i.e.* $\mathfrak{d}\beta_i$ is conserved
- β_i is a current variationally equivalent to the canonical strong Noether current $\nu_i + \epsilon_i$

Second variational derivative and globality of variation of strong currents

- $j^r\Xi$ is a generalized symmetry: $\mathcal{L}_{j^r\Xi}\lambda_i$ is *closed*
- since the variational Lie derivative of closed classes trivializes cohomology classes,

$$\delta_{n-1}(\mathcal{L}_{j^r\Xi}\mathcal{L}_{j^r\Xi}\lambda_i) = 0 \implies \mathcal{L}_{j^r\Xi}\mathcal{L}_{j^r\Xi}\lambda_i = d_H\psi_i$$

therefore

$$d_H\mathcal{L}_{j^r\Xi}(\nu_i + \epsilon_i) = d_H\psi_i$$

with $\mathfrak{d}\psi_i = 0$ (slight abuse of notation: here we mean that $\mathfrak{d}\psi_i = \mathfrak{d}d_H\zeta_i$, *i.e.* ψ_i can be added of a horizontal differential of a local object so that $\psi_i - d_H\zeta_i$ is the restriction of a global current $\bar{\psi}$)

Second variational derivative and globality of variation of strong currents

- thus, as a consequence of the invariance of field equations, the *variation of the strong Noether current* $\mathcal{L}_{j^r\Xi}(\nu_i + \epsilon_i)$ is variationally equivalent to a global current
- we characterize more precisely this fact: under certain conditions a global representative is given by the *canonical* Noether current associated with the 'deformed' Lagrangian $\mathcal{L}_{j^r\Xi}\lambda_i$

A (global) representative

Theorem

The variation of a local strong Noether current by a generalized symmetry is variationally equivalent to the canonical Noether current associated to the variation of the local Lagrangians by the same symmetry, i.e. *we have*

$$\mathcal{L}_{j^r\Xi}(\nu_i + \epsilon_i) \simeq \epsilon_{\mathcal{L}_{j^r\Xi}\lambda_i}$$

indeed, let $\gamma_{\phi_i} =_{loc.} d_H\phi_i$ be a (global) locally variationally trivial Lagrangian

- we have

$$d_H\epsilon_{\gamma_{\phi_i}} = \mathcal{L}_{j^r\Xi}\gamma_{\phi_i} \doteq \gamma_{\mathcal{L}_{j^r\Xi}\phi_i} = d_H(\mathcal{L}_{j^r\Xi}\phi_i)$$

- by applying the above to $\gamma =_{loc.} d_H(\nu_i + \epsilon_i)$ we get the statement

Sufficient condition for the globality

Corollary

- *if, in particular, the coboundary of* $\phi_i \simeq \nu_i + \epsilon_i$ *is locally exact* ($\iff \mathfrak{d}\mathcal{L}_{j^r\Xi}\lambda_i = 0$) , *we have* $0 = \delta_{n-1}(\mathcal{L}_{j^r\Xi}\gamma_{\phi_i}) = \delta_{n-1}(\gamma_{\mathcal{L}_{j^r\Xi}\phi_i})$
- i.e. $\mathcal{L}_{j^r\Xi}\phi_i$ *is equivalent to a global current and in particular* $\mathcal{L}_{j^r\Xi}\phi_i \simeq \epsilon_\gamma$, i.e. *the (local) current* $\mathcal{L}_{j^r\Xi}\phi_i$ *is variationally equivalent to the global current* ϵ_γ

Conservation laws from the vanishing of the second variational derivative

Theorem

- *let Ξ be a generalized symmetry of the Euler–Lagrange form η_λ*
- *if the second variational derivative by the generalized symmetry of the local problems is vanishing, we have the conservation law* $d_H \mathcal{L}_\Xi(\nu_i + \epsilon_i) = 0$

- this means that the current $\epsilon_{\mathcal{L}_{j^r\Xi}\lambda_i}$ is also conserved and the corresponding conservation laws reads $d_H(\epsilon_{\mathcal{L}_{j^r\Xi}\lambda_i}) = 0$
- this canonical Noether current is conserved along *any section*
- from the theorem above and the previous example, we get **Corollary**: for the variation of the canonical Noether current ϵ_{λ_i}, we have the strong conservation law

$$\mathcal{L}_{j^r\Xi}\epsilon_{\lambda_i} \simeq \epsilon_{d_H \epsilon_{\lambda_i}}$$