Topological obstructions in Lagrangian field theories

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ABSTRACT

We relate the existence of Noether global conserved currents associated with locally variational field equations to existence of global solutions for a local variational problem generating global equations. Both can be characterized as the vanishing of certain cohomology classes.

We suggest a parallelism between the geometric interpretation of characteristic classes as obstruction to the existence of flat principal connections and the interpretation of certain de Rham cohomology classes to be the obstruction to the existence of global extremals for a local variational principle.

Framework

- the r-th order prolongation of a fibered manifold $\pi : \mathbf{Y} \to \mathbf{X}$, with $\dim \mathbf{X} = n$ and $\dim \mathbf{Y} = n + m$ is the configuration space.
- physical fields and their partial rth order derivatives are (local) sections of $\pi^r : J_r\mathbf{Y} \to \mathbf{X}$.
- the affine bundle structure $\pi^{r+1}_r : J_{r+1}\mathbf{Y} \to J_r\mathbf{Y}$ induce the splitting

$$J_r\mathbf{Y} \times_{J_{r-1}\mathbf{Y}} T^*J_{r-1}\mathbf{Y} = J_r\mathbf{Y} \times_{J_{r-1}\mathbf{Y}} (T^*\mathbf{X} \oplus V^*J_{r-1}\mathbf{Y})$$

- *splittings* in horizontal and vertical parts of objetcts on $J_r\mathbf{Y}$.
- define sheaves of contact forms Θ^*_r by the kernel of the horizontalization.
- The sheaves Θ^*_r form an exact subsequence of the de Rham sequence on $J_r\mathbf{Y}$
- define the quotient sequence $\{0 \to I\!R_{\mathbf{Y}} \to \Lambda^*_r/\Theta^*_r, \mathcal{E}_*\}$, the r–th order variational sequence on $\mathbf{Y} \to \mathbf{X}$
- it is an acyclic sheaf resolution of the constant sheaf $I\!R_{\mathbf{Y}}$.
- the cohomology of the complex of global sections $H^*_{VS}(\mathbf{Y})$ is naturally isomorphic to both the Čech cohomology $H^*(\mathbf{Y}, I\!R_{\mathbf{Y}})$ and the de Rham cohomology $H^*_{dR}(\mathbf{Y})$.
- Lagrangians are sheaf sections $\lambda \in \mathcal{V}^n_r$, while $\mathcal{E}_n$ is called the Euler-Lagrange morphism and $\mathcal{E}_{n+1}$ the Helmholtz morphism.
- Euler-Lagrange equations given by $\mathcal{E}_n(\lambda) \circ j_{2r+1}\sigma = 0$ for (local) sections $\sigma : \mathbf{X} \to \mathbf{Y}$.
- sections $\eta \in \mathcal{V}^{n+1}_r$ are called *source or dynamical forms*,
- nontrivial cohomology of $\mathbf{Y}$: given a closed section of a quotient sheaf of the variational sequence, when is this section also globally exact?
- answer: let $\mathbf{K}^p_r \doteq Ker\, \mathcal{E}_p$ we have a natural short exact sequence of sheaves inducing a long exact sequence in Čech cohomology
- the *connecting homomorphism* $\delta_p = i^{-1} \circ \mathfrak{d} \circ \mathcal{E}^{-1}_p$, is the mapping of cohomologies in the corresponding diagram of cochain complexes ($\mathfrak{d}$ is the *Čech coboundary operator*).
- every global section $\eta \in \mathcal{E}_p(\mathcal{V}^p_r)$, *i.e.* locally variational, defines a cohomology class $\delta_p(\eta) \in H^1(\mathbf{Y}, \mathbf{K}^p_r) \simeq H^{p+1}_{VS}(\mathbf{Y}) \simeq H^{p+1}_{dR}(\mathbf{Y})$.
- every non vanishing cohomology class in $H^p_{dR}(\mathbf{Y})$ gives rise to local variational problems.
- it is clear that η is globally variational if and only if $\delta_p(\eta) = 0$.

Main results

- We obtain a general statement (*Theorem 1.*) saying that if a variational problem admits global critical sections then all conservation laws derived from symmetries of the field equations admit global conserved quantities.

This result relays on the relation between globality of conserved currents associated with invariance of (locally) variational field equations and certain cohomology classes defined by the dynamical forms themselves and their symmetries; such classes are isomorphic to certain de Rham cohomology classes of the space of fields. We relate such classes to cohomology classes of the basis manifold of the theory by pull-back *via* global sections.

- If one of such sections defines a non vanishing cohomology class, then there is no global critical section in its homotopy class (*Corollary 1.*). This is strictly based on the 'modulo contact' structure of the calculus of variations on fibered manifolds.
- we prove that in the case of a (closed) 3-dimensional Chern–Simons gauge theory, the cohomological obstruction to the existence of global solutions is sharp and equivalent to the usual obstruction in terms of the Chern characteristic class for the flatness of a principal connection (*Theorem 2.*).

3D Chern-Simons theories

Theorem 1. Let η be the dynamical form of a local variational problem and let $H^n_{dR}(\mathbf{Y}) \sim \pi^*(H^n_{dR}(\mathbf{X}))$. If the variational problem admits (global) critical sections then all conservation laws derived from symmetries of the global field equations admit global conserved quantities.

Corollary 1. Let σ be a section of $\mathbf{Y}$ over $\mathbf{X}$. If $0 \neq j\sigma^*([\Xi_V \rfloor \eta]) \in H^n(\mathbf{X})$, then there is no (global) critical section in the homotopy class of σ.

Under the condition $H^n_{dR}(\mathbf{Y}) \sim \pi^*(H^n_{dR}(\mathbf{X}))$, as in the previous theorem, there are no (global) critical sections if $[\Xi_V \rfloor \eta] \neq 0$.

Theorem 2. Chern–Simons theories on closed, *i.e.* compact without boundary, 3-manifolds admit global critical sections if and only if $[\Xi \rfloor (\pi^1_0)^*(\Omega_2(\mathcal{F}))] = 0$.

Conclusions

- under certain (not much restrictive) conditions, if a variational problem admits (global) critical sections then all conservation laws derived from symmetries of the field equations admit global conserved quantities.
- we have identified an obstruction to the existence of global solutions for the variational problem defined by a given dynamical form.
- there is no (global) critical section in the homotopy class of each global section defining a non vanishing cohomology class.
- purely gauge 3D Chern–Simons theory: the variational obstruction is sharp *i.e.* equivalent to the first Chern characteristic class.

Perspective

- different Chern–Simons terms in theories of topological gravity share being some kind of 'transgression' for some kind of 'curvature'.
- the choice of the form of Chern–Simons terms: obstruction for the existence of global critical sections *i.e.* of global Noether–Bessel-Hagen currents.
- understand if and how solutions of equations of motions are affected by adding transgression terms such that there exist global critical sections.
- the vanishing of added dynamical terms (some kind of 'curvature') along solutions grants that the corresponding obstruction to globality of conserved quantities vanishes. This is *e.g.* the case of the Cotton tensor in topologically massive gauge theories which vanishes along Einstein solutions; or if the 'curvature' terms added to the equations are identically vanishing (sufficient although not necessary condition)
- the necessary condition is ultimately given by Corollary 1.

References

[1] M. Palese, E. Winterroth, *Journal of Mathematical Physics* **58** 023502 (2017)