Galilean electrodynamics and duality in the galilean limit

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Abstract

This poster examines the non-relativistic limits of Maxwell electrodynamics, with particular attention to the fate of electromagnetic duality under the contraction ($c \to \infty$). This work in progress explores whether, and in what precise sense, the Galilean electric and magnetic sectors may be regarded as dual descriptions. In their standard potential formulations, neither sector follows directly from a local Lagrangian without additional fields or a non-local reformulation. Consequently, the usual parent-action argument alone does not settle the question. A remnant of the duality group can be constructed by an appropriate scaling of the angle parameter.

Relativistic duality and the Galilean limits

In vacuum and in units $c = 1$, Maxwell's equations are

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0,$$
$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} - \partial_t \mathbf{E} = 0.$$

Their form is unchanged under the discrete electric–magnetic interchange

$$\mathbf{E} \longmapsto \mathbf{B}, \qquad \mathbf{B} \longmapsto -\mathbf{E}.$$

More generally, this extends to the continuous $SO(2)$ duality rotation

$$\begin{pmatrix} \mathbf{E}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}.$$

Le Bellac and Lévy-Leblond showed that Maxwell theory admits two inequivalent Galilean regimes:

$$\nabla \cdot \mathbf{E}_{\mathrm{e}} = \rho_{\mathrm{e}}, \qquad \nabla \times \mathbf{E}_{\mathrm{e}} = 0,$$
$$\nabla \cdot \mathbf{B}_{\mathrm{e}} = 0, \qquad \nabla \times \mathbf{B}_{\mathrm{e}} - \partial_t \mathbf{E}_{\mathrm{e}} = \mathbf{J}_{\mathrm{e}} \qquad \text{Electric limit, } |\mathbf{E}| \gg c|\mathbf{B}|$$

$$\nabla \cdot \mathbf{E}_{\mathrm{m}} = \rho_{\mathrm{m}}, \qquad \nabla \times \mathbf{E}_{\mathrm{m}} + \partial_t \mathbf{B}_{\mathrm{m}} = 0,$$
$$\nabla \cdot \mathbf{B}_{\mathrm{m}} = 0, \qquad \nabla \times \mathbf{B}_{\mathrm{m}} = \mathbf{J}_{\mathrm{m}} \qquad \text{Magnetic limit, , } |\mathbf{E}| \ll c|\mathbf{B}|$$

Each limit retains a different Maxwell time-derivative term. The relativistic duality rotation, however, mixes precisely the two fields whose relative scale is important for the limit. What becomes of the continuous duality group in a controlled $c \to \infty$ contraction?

Related approaches

- **Covariant contractions:** different scalings of relativistic potential components produce the electric and magnetic sectors; covariant and contravariant potentials become independent in Galilean geometry [1, 2].
- **Galilean conformal symmetry:** the Helmholtz analysis yields an extended action for the magnetic sector, but not for the electric sector within the field extensions considered [3].
- **Parent actions:** an enlarged first-order formulation relates the two sectors off shell, although each reduced action misses one divergence constraint [4].

References

Completely non-exhaustive list

[1] A. Bagchi, R. Basu and A. Mehra, *Galilean Conformal Electrodynamics*, JHEP **11** (2014) 061.

[2] A. Mehra and Y. Sanghavi, *Galilean Electrodynamics: Covariant Formulation and Lagrangian*, JHEP **09** (2021) 078.

[3] K. Banerjee, R. Basu and A. Mohan, *Uniqueness of Galilean Conformal Electrodynamics*, JHEP **11** (2019) 041.

[4] J.A. O'Connor and S. Pekar, *A Note on Non-Lorentzian Duality Symmetries*, JHEP **03** (2025) 084.

[5] G. Manfredi, *Non-relativistic limits of Maxwell's equations*, Eur. J. Phys. **34** (2013) 859.

[6] J.A. Heras, *The Galilean limits of Maxwell's equations*, Am. J. Phys. **78** (2010) 1048.

Why scaling matters: Manfredi's approach

Taking $c \to \infty$ is ambiguous until one specifies what is held fixed. Explicit powers of c depend on the system of units, while additional factors are hidden in the relative normalizations of $\mathbf{E}$ and $\mathbf{B}$, and of ρ and $\mathbf{J}$.

Manfredi [5] makes these assumptions explicit by introducing reference scales

$$L,\ T,\ \overline{E},\ \overline{B},\ \overline{\rho},\ \overline{J}, \qquad V = \frac{L}{T}, \quad \overline{E} = c\,\overline{B}, \quad \overline{J} = V\,\overline{\rho}.$$

The dimensionless Maxwell equations then depend on

$$\beta = \frac{V}{c}, \qquad \alpha = \frac{\overline{E}\,\varepsilon_0}{\overline{\rho}\,L}.$$

The electric and magnetic limits correspond respectively to

$$\beta \to 0, \quad \alpha = O(1), \qquad \text{and} \qquad \alpha, \beta \to 0, \quad \frac{\alpha}{\beta} = O(1).$$

This provides a systematic limiting procedure, but at the cost of introducing several external reference scales. Can the relevant c-dependence instead be exposed directly within Maxwell's equations?

Separating units from propagation: Heras

Heras [6] writes the unit-independent static equations

$$\nabla \cdot \mathbf{E} = \alpha\rho, \qquad \nabla \times \mathbf{E} = 0,$$
$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = \beta\mathbf{J}.$$

From magnitudes of forces between charges and wires, where an additional constant χ relates to the choice of units, they realize the constants are related by

$$c_u^2 = \frac{\alpha}{\beta\chi}.$$

Importantly, c_u follows from static electric and magnetic measurements, before any propagation law is introduced.

Requiring a consistent dynamical extension gives

$$\nabla \cdot \mathbf{E} = \alpha\rho,$$
$$\nabla \times \mathbf{E} + \chi\frac{c_u^2}{c^2}\partial_t\mathbf{B} = 0,$$
$$\nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{B} - \frac{1}{\chi c_u^2}\partial_t\mathbf{E} = \beta\mathbf{J}.$$

Here c is the propagation speed, whereas c_u retains the unit-dependent normalization of the fields. Their observed equality, $c = c_u$, recovers ordinary Maxwell theory.

This separation makes the instantaneous contraction unambiguous:

$$c \to \infty \qquad \text{with} \qquad c_u, \alpha, \beta, \chi \text{ fixed.}$$

The freedom to use SI, Gaussian or Heaviside–Lorentz units remains manifest throughout [6].

A one-parameter family of contractions

Introduce

$$\varepsilon \equiv \frac{c_u}{c}, \varepsilon \to 0,$$

and define fields and sources that remain finite in the limit:

$$\mathbf{E}_q = \varepsilon^{-q}\mathbf{E}, \qquad \mathbf{B}_q = \varepsilon^{q}\mathbf{B}, \qquad \rho_q = \varepsilon^{-q}\rho, \qquad \mathbf{J}_q = \varepsilon^{q}\mathbf{J}.$$

Heras's equations become

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{E}_q &= \alpha\rho_q, \\ \nabla \times \mathbf{E}_q + \chi\varepsilon^{2-2q}\partial_t\mathbf{B}_q &= 0, \\ \nabla \cdot \mathbf{B}_q &= 0, \\ \nabla \times \mathbf{B}_q - \frac{\varepsilon^{2q}}{\chi c_u^2}\partial_t\mathbf{E}_q &= \beta\mathbf{J}_q. \end{aligned}}$$

The two time-derivative terms are therefore controlled by complementary powers of the same limiting parameter:

q	Faraday	Ampère	$\varepsilon \to 0$ limit
$q < 0$	0	divergent	singular
$q = 0$	0	finite	electric / Heras instantaneous
$0 < q < 1$	0	0	static family
$q = 1$	finite	0	magnetic
$q > 1$	divergent	0	singular

Thus the electric and magnetic theories are the endpoints $q = 0$ and $q = 1$ of a common family. There is a consistent limit also in the interior.

What happens to electromagnetic duality?

In vacuum, the relativistic duality rotation expressed in the scaled fields is

$$\begin{pmatrix} \mathbf{E}'_q \\ \mathbf{B}'_q \end{pmatrix} = \begin{pmatrix} \cos\theta & \chi c_u\,\varepsilon^{1-2q}\sin\theta \\ -\dfrac{1}{\chi c_u}\varepsilon^{2q-1}\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{E}_q \\ \mathbf{B}_q \end{pmatrix}.$$

For a fixed, nonzero θ, the limit $\varepsilon \to 0$ is generically singular: Only the balanced scaling $q = \frac{1}{2}$ retains a finite rotation A finite transformation can be forced for $q \neq \frac{1}{2}$ by also scaling the duality angle,

$$\theta = \varepsilon^{|2q-1|}\vartheta.$$

The rotation then contracts to a shear:

$$q < \tfrac{1}{2}: \ \mathbf{E}'_q = \mathbf{E}_q, \qquad \mathbf{B}'_q = \mathbf{B}_q - \frac{\vartheta}{\chi c_u}\mathbf{E}_q,$$
$$q > \tfrac{1}{2}: \ \mathbf{E}'_q = \mathbf{E}_q + \chi c_u\vartheta\,\mathbf{B}_q, \qquad \mathbf{B}'_q = \mathbf{B}_q.$$

Thus the relativistic $SO(2)$ rotation is not generically inherited by the Galilean limits. The surviving symmetry could be interpreted as a remnant of the relativistic rotation, but requires an additional, c-dependent scaling of the transformation parameter.