Connected Component Structure for Automorphism Groups of Weil Algebras and Its Applications

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Why connected components?

A *Weil algebra* is a finite-dimensional local commutative unital $\mathbb{R}$-algebra with unique maximal ideal $\mathfrak{n}$ satisfying

$$A/\mathfrak{n} \cong \mathbb{R}, \qquad \mathfrak{n}^{r+1} = 0, \qquad w(A) = \dim_{\mathbb{R}}(\mathfrak{n}/\mathfrak{n}^2).$$

Every Weil algebra has a presentation

$$A = D_n^r/I, \qquad D_n^r = \mathbb{R}[x_1, \dots, x_n]/\mathfrak{m}^{r+1},$$

where $n = w(A)$. A regular A-point $X : C^\infty(M) \twoheadrightarrow A$ determines the contact element

$$[X]_A = \mathrm{Aut}(A) \cdot X = \{\varphi \circ X : \varphi \in \mathrm{Aut}(A)\}.$$

The bundle of contact elements is

$$K^A M = \mathrm{reg}\, T^A M / \mathrm{Aut}(A).$$

Writing $G = \mathrm{Aut}(A)$ and denoting its identity component by G_0, the group

$$\boxed{\pi_0(G) = G/G_0}$$

records the reparametrizations that cannot be continuously deformed to the identity. It therefore controls the discrete orientation data retained by Weil contact elements.

From relations to components

For

$$A = D_n^r/I, \qquad D_n^r = \mathbb{R}[x_1, \dots, x_n]/\mathfrak{m}^{r+1},$$

the calculation is finite and algorithmic:

1. Reduce monomials modulo I and choose a basis $\{1, e_1, \dots, e_N\}$.
2. Write $\varphi(x_i) = \sum_{j=1}^N c_{ij} e_j$.
3. Impose $\varphi(f) = 0$ for every defining relation $f \in I$.
4. Require the induced linear map to be nonsingular:
$$M_1(\varphi) : \mathfrak{n}/\mathfrak{n}^2 \longrightarrow \mathfrak{n}/\mathfrak{n}^2, \qquad \det M_1(\varphi) \neq 0.$$
5. Identify sign invariants and construct paths inside the solution set.

> **Caution.** Different algebraic branches of the polynomial solution set need not be different connected components.

Every $\varphi \in \mathrm{Aut}(A)$ induces

$$\epsilon_A(\varphi) : \mathfrak{n}/\mathfrak{n}^2 \longrightarrow \mathfrak{n}/\mathfrak{n}^2, \qquad D_1(\varphi) = \det \epsilon_A(\varphi).$$

Theorem (Kureš–Šútora). If $\mathrm{Aut}(A)$ is connected, then

$$\boxed{D_1(\mathrm{Aut}\, A) = \{1\} \quad \text{or} \quad D_1(\mathrm{Aut}\, A) = (0, \infty).}$$

Worked example: the Drozd algebra

Let

$$B = D_2^2/\langle Y^2 \rangle, \qquad \mathcal{B} = (1, X, Y, X^2, 2XY).$$

Write

$$\varphi(X) = AX + BY + CX^2 + D(2XY),$$
$$\varphi(Y) = EX + FY + GX^2 + H(2XY).$$

Preservation of $Y^2 = 0$ gives

$$0 = \varphi(Y)^2 = E^2X^2 + EF(2XY), \qquad \text{hence } E = 0.$$

After omitting the fixed unit, an automorphism has matrix

$$M = \begin{pmatrix} A & B & C & D \\ 0 & F & G & H \\ 0 & 0 & A^2 & AB \\ 0 & 0 & 0 & AF \end{pmatrix}, \qquad AF \neq 0,$$

and its linear part on $\mathfrak{n}/\mathfrak{n}^2$ is

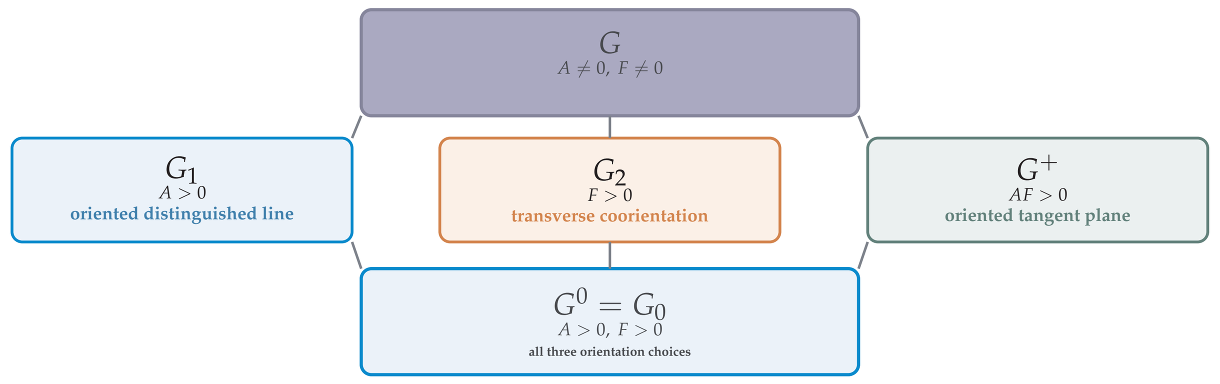
$$M_1 = \begin{pmatrix} A & B \\ 0 & F \end{pmatrix}.$$

The signs of A and F cannot change along a path, so

$$\boxed{\pi_0(\mathrm{Aut}\, B) \cong \mathbb{Z}_2 \times \mathbb{Z}_2}.$$

Thus $\mathrm{Aut}(B)$ has four components and exactly five open subgroups.

The five open subgroups of $G = \mathrm{Aut}(B)$



Determinant warning. Although $\det M = A^4F^2 > 0$, the two signs of A and F yield four components; $\det M_1 = AF$ sees only two classes.

Geometric meaning. A regular B-velocity in $\mathbb{R}^3$ records

$$Z = (\Gamma_s, \Gamma_t, \Gamma_{ss}, \Gamma_{st}), \qquad \mathrm{rank}(\Gamma_s, \Gamma_t) = 2,$$

and is represented to second order by

$$\Gamma(s,t) = P + su + tv + \frac{1}{2}s^2 a + stb + O(3).$$

The relation $Y^2 = 0$ removes the t^2 term. In coordinates with horizontal tangent plane, the normal part is

$$z = \frac{1}{2}\alpha s^2 + \beta st.$$

Thus $\beta \neq 0$ gives a hyperbolic paraboloid; $\beta = 0$, $\alpha \neq 0$ gives a parabolic cylinder; and $\alpha = \beta = 0$ gives a plane, to second order.

A high-order example: branches versus components

Consider the non-homogeneous Weil algebra

$$A_c = D_2^{10}/\langle x^4 + y^7,\ x^5 + y^8 \rangle.$$

Reduction of the 66 monomials gives $\dim A_c = 32$. Choose a reduced basis $(1, e_1, \dots, e_{31})$ and write

$$\varphi(x) = \sum_{j=1}^{31} A_j e_j, \qquad \varphi(y) = \sum_{j=1}^{31} B_j e_j.$$

Thus A_j, B_j are coefficient coordinates on $\mathrm{Aut}(A_c)$, not functions. The first relations force the linear part to be the identity:

$$A_1 = B_2 = 1, \qquad A_2 = B_1 = 0.$$

For brevity set

$$G = A_6 + A_7 + A_8, \quad H = 4\sum_{j=10}^{13} A_j, \quad K = 7\sum_{j=10}^{13} B_j.$$

The remaining nonlinear relation is

$$\frac{9}{8}G(2B_4 - A_3) = K - H,$$

which yields two algebraic types:

(i) $\quad G = 0,\ K = H,$

(ii) $\quad G \neq 0,\ \ 2B_4 - A_3 = \dfrac{8(K-H)}{9G}.$

There are 62 initial coefficients. The zero-monomial relations impose 3 linear conditions, the remaining A-relations impose 5, and the B-relations impose 6. The nonlinear equation contributes one more. Hence

$$\boxed{\dim \mathrm{Aut}(A_c) = 62 - 14 - 1 = 47.}$$

This is a *dimension*, not a number of components. The type-(i) locus has dimension $62 - 14 - 2 = 46$.

How curves determine components. After solving the relations, regard $\mathrm{Aut}(A)$ as a subset of a finite-dimensional coefficient space. Since it is a Lie group, its connected components are path-connected. A continuous nonzero coefficient or minor has constant sign along a path; conversely, an explicit curve inside the solution set proves that its endpoints are in the same component.
The two types above are *not* different components. For a point of type (ii), write

$$P^0 = (G^0, K^0, H^0, B_4^0, A_3^0)$$

and define the coordinate functions of one curve $\Gamma(t)$ by

$$g(t) = (1-t)G^0, \qquad h(t) = H^0,$$
$$k(t) = H^0 + (1-t)(K^0 - H^0),$$
$$b_4(t) = B_4^0, \qquad a_3(t) = A_3^0.$$

The superscript 0 denotes the starting value; the lowercase symbols are the coordinates of $\Gamma(t)$. Since P^0 satisfies the nonlinear relation,

$$\frac{9}{8}g(t)\big(2b_4(t) - a_3(t)\big) = k(t) - h(t)$$

for every $t \in [0,1]$. At $t = 1$, $g(1) = 0$ and $k(1) = h(1)$, so the curve reaches type (i). The remaining free coefficients can then be contracted linearly inside the affine type-(i) locus to the identity.
Therefore

$$\boxed{\mathrm{Aut}(A_c) = G_{A_c} = U_{A_c} \quad \text{is connected.}}$$

Here $G_{A_c} = (\mathrm{Aut}\, A_c)_0$, $U_{A_c} = \ker \epsilon_{A_c}$, and $\mathrm{Fix}(A_c) \supseteq \mathbb{R}x^4y^3$. Thus **polynomial branches must be tested by explicit paths** before they are declared to be components.

Applications to Weil contact elements

General construction. A regular A-point is a surjective algebra homomorphism $p^A : C^\infty(M) \twoheadrightarrow A$. The group $G = \mathrm{Aut}(A)$ acts by postcomposition. Quotienting first by its identity component gives

$$\widetilde{K}^A M := \mathrm{reg}\, T^A M/G_0 \longrightarrow K^A M := \mathrm{reg}\, T^A M/G,$$

whose discrete fibre is $\pi_0(G) = G/G_0$. For $A = D_n^r$ one has $G \cong G_n^r$, the classical jet group. If $G = G_0$, the oriented and full quotients coincide.

Classical surface case. For $A = D_2^2$ and an immersed surface germ $\Gamma : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, P)$, a second-order velocity records

$$Y = (\Gamma_s, \Gamma_t, \Gamma_{ss}, \Gamma_{st}, \Gamma_{tt}), \qquad \mathrm{rank}(\Gamma_s, \Gamma_t) = 2.$$

The full jet group forgets the source coordinates and retains

$$(P, \Pi, \mathrm{II}^\perp),$$

whereas its identity component preserves the source orientation and yields

$$(P, N, \mathrm{II}).$$

Changing the orientation acts by

$$(P, N, \mathrm{II}) \longmapsto (P, -N, -\mathrm{II}).$$

Thus the familiar unit-normal choice for surfaces is the simplest instance of the component-group mechanism.