Towards General Relativity as a generalized Yang-Mills theory

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(joint work with Lorenzo Fatibene; based on [4], [5] and [7])

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Can General Relativity be seen as a gauge theory?

A positive answer to this question appeared already in [6] by which time **principal bundles** and **variational calculus on fiber bundles** had been identified as the right mathematical framework for gauge theories. This particular issue still provides inspiration for developments both in mathematics and in physics; see for example the recent paper [2] on teleparallel gravity.

We consider the problem from a different angle: we want to find **more general scheme** of which both usual principal bundle theory and General Relativity can be seen as specializations.

Introduction

How far should we go in the enlargement of the fundamental geometric structure? A promising choice is to consider the notion of **generalized principal bundle** as proposed by Castrillón López and Rodríguez Abella in [3].

Aim

Developing an instance of **generalized gauge theories**, anticipating a unifying language for different kinds of covariant field theories, with an emphasis on Yang-Mills theories and General Relativity.

Generalized principal bundles

Firstly, we should consider:

Lie group fiber bundles

A **Lie group fiber bundle** with typical fiber a Lie group G is a fiber bundle $(\mathcal{G}, M, \pi_{\mathcal{G},M}, G)$ such that:
- $\forall\, x \in M$, the fiber $\mathcal{G}_x = (\pi_{\mathcal{G},M})^{-1}(x)$ is equipped with a Lie group structure.
- $\forall\, x \in M$, there is a local trivialization ψ preserving the Lie group structure fiberwise, i.e. $\psi|_{\mathcal{G}_x} \colon \mathcal{G}_x \to \{x\} \times G$ is a Lie group isomorphism.

As a consequence, there exist fibered coordinates (x^μ, g^I) on $(\mathcal{G}, M, \pi_{\mathcal{G},M}, G)$ with transformation laws satisfying unitarity and multiplicativity conditions.

We say that a **(right) fibered action** of a Lie group fiber bundle $(\mathcal{G}, M, \pi_{\mathcal{G},M}, G)$ on a fiber bundle $(P, M, \pi_{P,M}, F)$ is a bundle morphism $\Phi \colon P \times_M \mathcal{G} \to P$ over the identity id_M, such that:
- $\Phi(p, e_x) = p$, with $\pi_{P,M}(p) = x$, where e_x is the identity element of the Lie group $\mathcal{G}_x$.
- $\Phi\Big(p, \mathcal{M}(\gamma_1, \gamma_2)\Big) = \Phi\Big(\Phi(p, \gamma_1), \gamma_2\Big)$, where the **multiplication map** $\mathcal{M} \colon \mathcal{G} \times_M \mathcal{G} \to \mathcal{G}$ is defined by taking the product of elements in the fibers of $\mathcal{G}$.

It is also possible to easily adapt the notions of **free** and **proper** action to fibered actions.

Given a fibered action Φ of $(\mathcal{G}, M, \pi_{\mathcal{G},M}, G)$ on $(P, M, \pi_{P,M}, F)$, we introduce an equivalence relation $\sim_{\mathcal{G}}$ on P declaring that:

$$p_1 \sim_{\mathcal{G}} p_2 \iff \exists\, \gamma \in \mathcal{G} \,|\, \Phi(p_1, \gamma) = p_2$$

for $p_1, p_2 \in P$.

One can then define the quotient space $\mathcal{S} = P/\mathcal{G} = P/\!\sim_{\mathcal{G}}$ for which it holds that:

Generalized Quotient Manifold Theorem

Given a free and proper fibered action Φ of a Lie group fiber bundle $(\mathcal{G}, M, \pi_{\mathcal{G},M}, G)$ on a fiber bundle $(P, M, \pi_{P,M}, F)$, then $\mathcal{S}$ admits a unique smooth structure such that $(P, \mathcal{S}, \pi_{P,\mathcal{S}}, G)$ is a fiber bundle, called **generalized principal bundle**.

It turns out that principal bundles and Lie group fiber bundles can be seen as examples of generalized principal bundles.

We used the result above to establish on a generalized principal bundle $(P, \mathcal{S}, \pi_{P,\mathcal{S}}, G)$ **generalized principal bundle coordinates** $(x^\mu, \sigma^\Theta, g^I)$, i.e. fibered coordinates adapted to the generalized principal bundle structure, which have transformation laws:

$$\begin{cases} x'^\mu = x'^\mu(x) \\ \sigma'^\Theta = \Sigma^\Theta(x, \sigma) \\ g'^I = \pi^I\Big(\varphi(x, \sigma), G(x, g)\Big) \end{cases}$$

where π^I are the local expressions of the product in the Lie group G, $\varphi(x, \sigma) = \varphi^A(x, \sigma)$ are the local expressions of a function φ (characterizing the generalized principal bundle) and $G(x, g) = G^B(x, g)$, as well as $x'^\mu(x)$, come from the transformation laws of the fibered coordinates (x^μ, g^I), with $\mu \in \{1, \dots, \dim(M)\}$, $\Theta \in \{1, \dots, \dim(F) - \dim(G)\}$, $A, B, I \in \{1, \dots, \dim(G)\}$.

Lie group fiber bundle connections and generalized principal connections

Generalized principal connections

Let η be a connection on $(\mathcal{G}, M, \pi_{\mathcal{G},M}, G)$. A **generalized principal connection** on $(P, \mathcal{S}, \pi_{P,\mathcal{S}}, G)$ associated to η is a connection τ, given by the collection of the horizontal lifts $\tau_p \colon T_{[p]_{\mathcal{G}}}(\mathcal{S}) \longrightarrow T_p(P)$, with $\pi_{P,\mathcal{S}}(p) = [p]_{\mathcal{G}}$, such that:

$$\tau_{\Phi(p,\gamma)} = T_{(p,\gamma)}(\Phi)\Big(\tau_p(\cdot), \eta_\gamma\Big(T_{[p]_{\mathcal{G}}}(\pi_{\mathcal{S},M})(\cdot)\Big)\Big),\ \forall\, (p, \gamma) \in P \times_M \mathcal{G}$$

It holds that the connection η is actually forced to be a **Lie group fiber bundle connection**, i.e. in fibered coordinates (x^μ, g^I) its connection coefficients $\eta^I_\mu(x, g)$ satisfy the conditions:
- $\eta^I_\mu(x, e) = 0$, where e is the identity element of the Lie group G.
- $\forall\, g, h \in G$, $\eta^I_\mu\Big(x, \pi(g, h)\Big) = \eta^A_\mu(x, g)\partial^1_A \pi^I(g, h) + \eta^A_\mu(x, h)\partial^2_A \pi^I(g, h)$, where $\partial^1_A \pi^I(g, h)$, $\partial^2_A \pi^I(g, h)$ are the partial derivatives regarding g^A and h^A, respectively.

In generalized principal bundle coordinates $(x^\mu, \sigma^\Theta, g^I)$ a generalized principal connection can be locally written with connection coefficients $A^I_\mu(x, \sigma, g)$ and $A^I_\Theta(x, \sigma, g)$ satisfying the conditions:
- $\forall\, g, h \in G$, $A^I_\mu\Big(x, \sigma, \pi(g, h)\Big) = A^A_\mu(x, \sigma, g)\partial^1_A \pi^I(g, h) + \eta^A_\mu(x, h)\partial^2_A \pi^I(g, h)$, where $\eta^A_\mu(x, h)$ are the connection coefficients of the fixed Lie group fiber bundle connection.
- $\forall\, g, h \in G$, $A^I_\Theta\Big(x, \sigma, \pi(g, h)\Big) = A^A_\Theta(x, \sigma, g)\partial^1_A \pi^I(g, h)$.

Generalized Yang-Mills theories

Yang-Mills theories are a special class of classical field theories, where the configuration bundle is the principal connection bundle $\mathrm{Con}(P)$, associated with a principal bundle $(P, S, \pi_{P,S}, G)$. Regarding the Lagrangian, we fix on S a metric g and we consider the non-degenerate Cartan-Killing bilinear form K on the Lie algebra $\mathfrak{g}$ (G semisimple). We then choose the **Yang-Mills Lagrangian** $L_{YM} = -\frac{1}{4} F^A_{\mu\nu} g^{\mu\alpha}(x) g^{\nu\beta}(x) F^B_{\alpha\beta} K_{AB} \sqrt{g(x)}ds$, where $F^A_{\mu\nu}$ are the principal connection curvature coefficients, $g(x) = \left|\det\Big(g_{\mu\nu}(x)\Big)\right|$ and $ds = dx^1 \wedge \dots \wedge dx^k$, where $k = \dim(S)$.

Using generalized principal bundles instead of principal bundles, it is possible to build a **generalized principal connection bundle** and to fix on it a **generalized Yang-Mills Lagrangian** which has the same algebraic structure of L_{YM}, but the $F^A_{\mu\nu}$ terms correspond to the generalized principal connection curvature coefficients. These choices lead to a notion of **generalized Yang-Mills theory**, which turns out to eventually include Yang-Mills theories.

Hints of a (generalized) gauge theory of gravity

Any vector bundle $(E, M, \pi_{E,M}, \mathbb{R}^l)$ is a generalized principal bundle and a generalized principal connection on it is an **affine connection** with generalized principal connection coefficients satisfying $A^I_\mu(x, v) = \sigma^I_\mu(x) + \eta^I_{A\mu}(x)v^A$, once set $\sigma^I_\mu(x) = A^I_\mu(x, 0)$ and where $\eta^I_{A\mu}(x)$ are the coefficients of a linear connection η, with respect to vector bundle coordinates (x^μ, v^I).

It turns out that $\sigma^I_\mu(x)$ are the local expressions of a **basic soldering form** σ on the vector bundle $(E, M, \pi_{E,M}, \mathbb{R}^l)$, i.e. a section of the vector bundle $(T^*(M) \otimes_M E, M, \pi_{T^*(M)\otimes_M E,M}, \mathbb{R}^{m \cdot l})$, where $m = \dim(M)$:

$$\sigma \colon M \longrightarrow T^*(M) \otimes_M E \colon x^\mu \longmapsto \Big(x^\mu, \sigma^I_\mu(x)\Big)$$

In this case the generalized principal connection curvature coefficients are of the form $F^A_{\mu\nu}(x) = (T_{\eta,\sigma})^B_{\mu\nu}(x)\bar{T}^A_B$, where $(T_{\eta,\sigma})^B_{\mu\nu}(x) = \partial_\mu \sigma^B_\nu(x) - \partial_\nu \sigma^B_\mu(x) + \sigma^C_\nu(x)\eta^B_{C\mu}(x) - \sigma^C_\mu(x)\eta^B_{C\nu}(x)$ are the **torsion coefficients** of the linear connection η with respect to the basic soldering form σ.

When the generalized principal bundle $(P, \mathcal{S}, \pi_{P,\mathcal{S}}, G)$ is set to be a vector bundle $(E, M, \pi_{E,M}, \mathbb{R}^l)$ over a Lorentzian manifold M with $\dim(M) = 4$, we want to show that it is possible to recover the **Vielbein** formulation of (vacuum) General Relativity in this setting.

If $\det\Big(\sigma^I_\mu(x)\Big) \neq 0$, which implies that $l = 4$, the present framework becomes the **fake tangent bundle** environment used in the definition of the Vielbein formalism, the Palatini formalism and even the teleparallel equivalent of General Relativity (see for example [1]). In this context, the basic soldering form σ is called Vielbein or **coframe field**.

This time we choose as configuration bundle the fiber bundle $(T^*(M) \otimes_M E, M, \pi_{T^*(M)\otimes_M E,M}, \mathbb{R}^{16})$ of which the sections are in a one to one correspondence with basic soldering forms, as seen above. It has fibered coordinates (x^μ, σ^I_μ) inducing $(x^\mu, \sigma^I_\mu, \sigma^I_{\mu,\nu})$ as fibered coordinates on $J^1(T^*(M) \otimes_M E)$.

Now we consider a modification of the generalized Yang-Mills Lagrangian:

$$L_{\text{Vielbein}} = \Big(-\frac{1}{4} F^A_{\mu\nu} g^{\mu\alpha} g^{\nu\beta} F^B_{\alpha\beta} \eta_{AB} + R\Big)\sqrt{g}ds$$

where we have essentially added the Einstein-Hilbert term, while also requiring the metric to be derived from the Vielbein σ and assuming the Vielbein postulate for the connection, i.e.:
- $g^{\mu\nu} = g^{\mu\nu}(\sigma) = \sigma^\mu_A \eta^{AB} \sigma^\nu_B$ and $\sqrt{g} = |\sigma| = \left|\det(\sigma^I_\mu)\right|$.
- $F^A_{\mu\nu} = (T_{\eta(\sigma),\sigma})^B_{\mu\nu}(\sigma)\bar{T}^A_B = \Big(\sigma^B_{\nu,\mu} - \sigma^B_{\mu,\nu} + \sigma^C_\nu \eta^B_{C\mu}(\sigma) - \sigma^C_\mu \eta^B_{C\nu}(\sigma)\Big)\bar{T}^A_B$, where $\eta^B_{C\mu}(\sigma) = \sigma^\alpha_C \eta^\beta_{\alpha\mu}(\sigma)\sigma^B_\beta - \sigma^\alpha_C \sigma^B_{\alpha,\mu}$ are the well-known **spin connection** coefficients. $\eta^\beta_{\alpha\mu}(\sigma)$ are just the Levi-Civita connection coefficients written in terms of the Vielbein σ.
- $R = R(\sigma)$ is the Ricci scalar written in terms of the Vielbein σ.

We can also readily observe that:

$$\begin{aligned}(T_{\eta(\sigma),\sigma})^B_{\mu\nu}(\sigma) &= \sigma^B_{\nu,\mu} - \sigma^B_{\mu,\nu} + \sigma^C_\nu \eta^B_{C\mu}(\sigma) - \sigma^C_\mu \eta^B_{C\nu}(\sigma) = \eta^\beta_{\nu\mu}(\sigma)\sigma^B_\beta - \eta^\beta_{\mu\nu}(\sigma)\sigma^B_\beta \\ &= \Big(\eta^\beta_{\nu\mu}(\sigma) - \eta^\beta_{\mu\nu}(\sigma)\Big)\sigma^B_\beta = 0\end{aligned}$$

since $\eta^\beta_{\alpha\mu}(\sigma)$ is the Levi-Civita connection, hence $F^A_{\mu\nu} = 0$. In this sense, we have that $L_{\text{Vielbein}} = R|\sigma|ds$ or, in other words, we have actually written the Hilbert-Einstein Lagrangian in the Vielbein formalism, reaching our intended aim.

Conclusion

The construction that we have presented highlights the flexibility of the generalized principal bundle theory and is the first evidence that Yang-Mills theories and General Relativity might be both seen as theories for a generalized principal connection, as specializations in the direction of standard principal bundles and of vector bundles, respectively.

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